

# THE SZEGÖ KERNEL FOR NON-PSEUDOCONVEX TUBE DOMAINS IN $\mathbb{C}^2$

MICHAEL GILLIAM AND JENNIFER HALFPAP

ABSTRACT. We consider the Szegő kernel associated with domains  $\Omega$  in  $\mathbb{C}^2$  given by

$$\Omega = \{ (z, w) : \operatorname{Im} w > b(\operatorname{Re} z) \}$$

for  $b$  a *non-convex* polynomial of even degree with positive leading coefficient. Such domains are not pseudoconvex. We give a precise description of a subset of  $\overline{\Omega} \times \overline{\Omega}$  on which the kernel and all of its derivatives are finite. We show, in particular, that for such domains, the Szegő kernel has singularities off the diagonal of  $\partial\Omega \times \partial\Omega$  as well as points on the diagonal at which it is finite.

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## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{C}^n$  be a domain with smooth boundary  $\partial\Omega$ , and let  $\mathcal{O}(\Omega)$  denote the space of holomorphic functions on  $\Omega$ . Associated with such a domain are certain operators: the Bergman projection  $\mathcal{B}$  and the Szegő projection  $\mathcal{S}$ . The former is the orthogonal projection of  $L^2(\Omega)$  onto the closed subspace  $L^2(\Omega) \cap \mathcal{O}(\Omega)$ , whereas the latter is the orthogonal projection of  $L^2(\partial\Omega)$  onto the closed subspace  $\mathcal{H}^2(\Omega)$  of boundary values of elements of  $\mathcal{O}(\Omega)$ . An important goal of research on these operators is to obtain results concerning their mapping properties (e.g., conditions under which they extend to bounded operators on the appropriate  $L^p$  spaces).

Often, understanding these operators begins with an investigation of the associated integral kernel; one identifies distributions  $B$  and  $S$  such that for  $f \in L^2(\Omega)$  and  $g \in L^2(\partial\Omega)$ ,

$$\begin{aligned} \mathcal{B}[f](z) &= \int_{\Omega} f(w) B(z, w) dw \\ \mathcal{S}[g](z) &= \int_{\partial\Omega} g(w) S(z, w) d\sigma(w). \end{aligned}$$

A first step in the analysis of these kernels is to describe the subset of  $\partial\Omega \times \partial\Omega$  to which they (and their derivatives) extend continuously. An early result of this sort is due to Kerzman [Ker72], who uses Kohn's formula to show that for  $\Omega \subseteq \mathbb{C}^n$  bounded and strongly pseudoconvex,  $B$  and its derivatives extend continuously to  $(\overline{\Omega} \times \overline{\Omega}) \setminus \Delta$ , where  $\Delta = \{ (z, w) \in \partial\Omega \times \partial\Omega : z = w \}$  is the diagonal of the boundary.

A further step in the analysis is to obtain sharp size estimates for these kernels and their derivatives near their singularities, together with mapping properties of the associated operators. This is done, for instance, by Nagel, Rosay, Stein, and Wainger [NRSW89] for finite-type domains in  $\mathbb{C}^2$  and by McNeal and Stein ([McN94], [MS94], [MS97]) for convex domains in  $\mathbb{C}^n$ .

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In contrast with the situation for pseudoconvex domains, comparatively little is known about the Szegő kernel for non-pseudoconvex domains, even in  $\mathbb{C}^2$ . Consider

$$(1.1) \quad \Omega = \{ (z_1 = x + iy, z_2 = t + i\xi) : \xi > b(x) \},$$

for a real-valued smooth function  $b$  satisfying  $\lim_{|x| \rightarrow \infty} b(x)/|x| = \infty$ . This domain is pseudoconvex precisely when  $b$  is convex. Some of the first results concerning the Szegő kernel in the non-pseudoconvex context are due to Carracino ([Car05], [Car07]). She obtains detailed estimates for the Szegő kernel on the boundary of a model domain of the type (1.1) with  $b$  a non-smooth, non-convex, piecewise quadratic function. She shows that the Szegő kernel has singularities off of  $\Delta$  in this case. Then in [GHar], the current authors identify a subset of  $\overline{\Omega} \times \overline{\Omega}$  on which the integrals defining the Szegő kernel and its derivatives are absolutely convergent for the case in which  $b$  is a non-convex quartic polynomial. In particular, this work shows that there are points *on the diagonal*  $\Delta$  at which the Szegő kernel is finite as well as points *off the diagonal* at which it is infinite.

In this paper, we explore this phenomenon in the much more general setting in which  $b$  is a non-convex even-degree polynomial with positive leading coefficient. Without loss of generality, we may suppose

$$(1.2) \quad b(x) = \frac{1}{2n}x^{2n} + \sum_{j=2}^{2n-1} a_j x^j, \quad n \geq 2.$$

Although the statements of the theorems in this paper closely resemble those in [GHar], the technical challenges in proving the theorems are rather different. We will comment on these substantial differences in due course.

We close this introductory section with a note on the motivations for studying *non-pseudoconvex* domains. To begin with, we are motivated by an interest in singular integral operators. The Szegő kernel for a pseudoconvex domain of finite type is an example of a non-isotropic smoothing operator; this is a class of operators that is well-understood. (See [NRSW89]). Carracino's work shows that the structure of the singularities of the Szegő kernel can be very different in the non-pseudoconvex setting, but it is inconclusive on the question of whether these kernels are related to flag kernels [NRS01] or product singular integral operators [NS04].

A second motivation for exploring the non-pseudoconvex setting arises because of its possible connection to the perhaps more natural problem of understanding the Szegő projection operator associated with a CR manifold of higher codimension such as the tube model

$$T_4 = \{ (z, w_2, w_3, w_4) : \operatorname{Re} w_j = [\operatorname{Re} z]^j \}.$$

One can derive an expression analogous to (2.4) for the kernel associated with the orthogonal projection of  $L^2(T_4)$  onto the subspace of functions annihilated by the tangential Cauchy-Riemann operators. The resulting integral is even more complicated in that setting; nonetheless, many of the technical challenges arising in that setting arise in the non-pseudoconvex setting as well. The analysis in the non-pseudoconvex setting can thus guide some of the analysis for higher-codimensional CR manifolds.

## 2. DEFINITIONS, NOTATION, AND STATEMENT OF RESULTS

We begin with a more precise discussion of the Szegő projection operator and its associated integral kernel for domains in  $\mathbb{C}^2$  having the form (1.1). We take  $b$  smooth so that  $\Omega \subset \mathbb{C}^2$  is smoothly-bounded. As above, let  $\mathcal{O}(\Omega)$  denote the space of functions holomorphic on  $\Omega$ . Define

$$\mathcal{H}^2(\Omega) := \left\{ F \in \mathcal{O}(\Omega) : \sup_{\varepsilon > 0} \int_{\partial\Omega} |F(x + iy, t + ib(x) + i\varepsilon)|^2 dx dy dt < \infty \right\}.$$

$\mathcal{H}^2(\Omega)$  can be identified with the set of all functions  $f$  in  $L^2(\partial\Omega)$  (which is itself identified with  $L^2(\mathbb{R}^3)$ ) which are solutions in the sense of distributions to

$$(2.1) \quad \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} - ib'(x) \frac{\partial}{\partial t} \right) [f] \equiv 0.$$

We define the *Szegő projection operator*  $\mathcal{S}$  to be the orthogonal projection of  $L^2(\partial\Omega)$  onto this (closed) subspace  $\mathcal{H}^2(\Omega)$ .

One establishes the existence of a unique integral kernel associated with the operator. This is discussed, for example, in [Ste72], where the approach is as follows: Begin with an orthonormal basis  $\{\phi_j\}$  for  $\mathcal{H}^2(\Omega)$  and form the sum

$$S(z, w) = \sum_{j=1}^{\infty} \phi_j(z) \overline{\phi_j(w)}.$$

One shows that this converges uniformly on compact subsets of  $\Omega \times \Omega$ , that  $\overline{S(z, \cdot)} \in \mathcal{H}^2(\Omega)$  for each  $z \in \Omega$ , and that for  $g \in \mathcal{H}^2(\Omega)$ ,

$$g(z) = \int_{\partial\Omega} S(z, w) g(w) d\sigma(w).$$

$S$  is then the *Szegő kernel*. From its construction it is clear that it will be smooth on  $\Omega \times \Omega$ . It may extend to a smooth function on some larger subset of  $\overline{\Omega} \times \overline{\Omega}$ .

For domains of the form (1.1), one can derive an explicit formula for the Szegő kernel. Let  $z = (z_1, z_2)$  and  $w = (w_1, w_2)$  be elements of  $\mathbb{C}^2$ . Set

$$(2.2) \quad N(\eta, \tau) = \int_{-\infty}^{\infty} e^{2\tau[\eta\lambda - b(\lambda)]} d\lambda.$$

Then

$$(2.3) \quad S(z, w) = c \iint_{\tau > 0} \tau e^{\eta\tau[z_1 + \bar{w}_1] + i\tau[z_2 - \bar{w}_2]} [N(\eta, \tau)]^{-1} d\eta d\tau,$$

where  $c$  is an absolute constant.

**Remark 2.1.** See [HNW10] for detailed discussions of  $\mathcal{H}^p$  spaces for unbounded domains, the derivations of such integral formulas, and the identification of  $\mathcal{H}^2(\Omega)$  with  $L^2(\partial\Omega)$  ( $= L^2(\mathbb{R}^3)$ ) functions satisfying the differential equation (2.1).

**Remark 2.2.** Many authors only consider  $S$  as a distribution on  $\partial\Omega \times \partial\Omega$  since  $S$  is smooth on  $\Omega \times \Omega$ . In this situation, one can identify the boundary with  $\mathbb{R}^3$  and consider the integral kernel

$$(2.4) \quad S[(x, y, t), (r, s, u)] = c \int_0^\infty \int_{-\infty}^\infty \tau e^{\tau[i(t-u) + i\eta(y-s) - [b(x) + b(r) - \eta(x+r)]]} [N(\eta, \tau)]^{-1} d\eta d\tau.$$

This is done, for example, in the work of Nagel [Nag86], Haslinger [Has95], and Carracino [Car05], [Car07].

We may now state our results:

Let  $b$  be as in (1.2). For each real  $\eta$ , set  $B_\eta(x) := -\eta x + b(x)$ . The set of minimizers of this function is of vital importance in our analysis. Thus we define

$$(2.5) \quad \Lambda_\eta = \{ \lambda : \inf_x B_\eta(x) = B_\eta(\lambda) \},$$

with  $\Lambda := \bigcup_\eta \Lambda_\eta$  and  $\mathcal{C} = \{ \eta : |\Lambda_\eta| > 1 \}$ . Furthermore, set

$$(2.6) \quad z = (z_1, z_2) = (x + iy, t + ib(x) + ih)$$

$$(2.7) \quad w = (w_1, w_2) = (r + is, u + ib(r) + ik),$$

and define

$$(2.8) \quad \Sigma = \{ (z, w) : x = r \text{ and } x \in \Lambda \} \cup \{ (z, w) : x, r \in \Lambda_c \text{ for } c \in \mathcal{C} \}.$$

Finally, for a function  $b$  continuous on  $\mathbb{R}$ , define the *Legendre transform* of  $b$  by

$$(2.9) \quad b^*(\eta) := \sup_{x \in \mathbb{R}} [\eta x - b(x)] = -\inf_x B_\eta(x).$$

**Theorem 2.3.** *The integral defining  $S(z, w)$  is absolutely convergent in the region in which*

$$(2.10) \quad h + k + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0.$$

*This is an open neighborhood of  $(\overline{\Omega} \times \overline{\Omega}) \setminus \Sigma$ . More generally, if  $i_1, j_1, i_2$ , and  $j_2$  are non-negative integers, then*

$$(2.11) \quad \partial_{z_1}^{i_1} \partial_{\bar{w}_1}^{j_1} \partial_{z_2}^{i_2} \partial_{\bar{w}_2}^{j_2} S(z, w) = c' \iint_{\tau > 0} e^{\eta\tau[z_1 + \bar{w}_1] + i\tau[z_2 - \bar{w}_2]} \frac{\eta^{i_1+j_1} \tau^{i_1+j_1+i_2+j_2+1}}{N(\eta, \tau)} d\eta d\tau$$

*is absolutely convergent in the same region.*

**Remark 2.4.** *Compare this with Theorem 3.2 in [HNW10] and with Theorem 2.3 in [GHar].*

**Theorem 2.5.** *If  $[(x + iy, t + ib(x)), (r + iy, t + ib(r))] \in \Sigma$ ,  $\mathcal{S}[(x, y, t), (r, y, t)]$  is infinite. Also, if  $\delta = h + k > 0$ ,*

$$\lim_{\delta \rightarrow 0^+} \mathcal{S}[(x + iy, t + i(b(x) + h)), (r + iy, t + i(b(r) + k))] = \infty.$$

We will show that the set  $\Sigma$  is equal to the diagonal  $\Delta$  of  $\partial\Omega \times \partial\Omega$  precisely when the polynomial  $b$  is convex. For non-convex  $b$ , there are both points *off the diagonal* that are contained in  $\Sigma$  and points *on the diagonal* that are not in  $\Sigma$ . We summarize this important observation in a corollary.

**Corollary 2.6.** *For tube domains (1.1) in  $\mathbb{C}^2$  with  $b$  an even-degree polynomial with positive leading coefficient, the Szegő kernel extends smoothly to  $(\overline{\Omega} \times \overline{\Omega}) \setminus \Delta$  if and only if  $b$  is convex.*

An analysis of the Szegő kernel begins with estimates of the integral  $N$  defined in (2.2). Observe that for fixed  $\eta \in \mathbb{R}$  and  $\tau > 0$ ,  $\lim_{|\lambda| \rightarrow \infty} 2\tau[\eta\lambda - b(\lambda)] = -2\tau \lim_{|\lambda| \rightarrow \infty} B_\eta(\lambda) = -\infty$ . The heuristic principle that guides the analysis of such integrals is that the main contribution comes from a neighborhood of the

point(s) at which the exponent attains its global maximum. For our integral  $N$ , let  $\lambda(\eta)$  denote the largest real number at which  $\inf_{\lambda} B_{\eta}(\lambda)$  is attained. Then

$$\begin{aligned} N(\eta, \tau) &= e^{-2\tau B_{\eta}(\lambda(\eta))} \int_{-\infty}^{\infty} e^{-2\tau[-\eta\lambda+b(\lambda)-B_{\eta}(\lambda(\eta))]} d\lambda \\ &= e^{2\tau b^*(\eta)} \int_{-\infty}^{\infty} e^{-2\tau p_{\eta}(\xi)} d\xi, \end{aligned}$$

where

$$(2.12) \quad p_{\eta}(\xi) := -\eta\xi + b(\xi + \lambda(\eta)) - b(\lambda(\eta))$$

is a non-negative polynomial vanishing to even order at the origin. Furthermore, by our choice of  $\lambda(\eta)$ , if for some  $\eta$ ,  $p_{\eta}(\xi) = 0$  for non-zero  $\xi$ , necessarily  $\xi < 0$ .

In Sections 3 and 4, we focus on understanding the main contribution to the integral  $N$  by exploring  $B_{\eta}$  and  $\lambda(\eta)$ , while in Section 5, we focus on estimating the integral that remains once we have taken out this main contribution. The theorems are established in Section 6.

### 3. GLOBAL PROPERTIES OF $\lambda(\eta)$ AND $B_{\eta}$

This section contains a number of technical lemmas on the long-term behavior of  $\lambda(\eta)$  and  $B_{\eta}(\lambda(\eta)) = -b^*(\eta)$ . Most of these results follow rather easily from the fact that  $B_{\eta}$  is a polynomial and  $\lambda(\eta)$  is one of its critical points.

**Lemma 3.1.**  $\lim_{\eta \rightarrow -\infty} \lambda(\eta) = -\infty$  and  $\lim_{\eta \rightarrow \infty} \lambda(\eta) = \infty$ . Furthermore,  $\lambda(\eta) \sim \eta^{\frac{1}{2n-1}}$  as  $|\eta| \rightarrow \infty$ .

*Proof.* We consider the case  $\eta \rightarrow -\infty$ . The case  $\eta \rightarrow \infty$  is established similarly.

Consider the equation  $b'(\omega) = \eta$ . Since  $b$  has even degree and positive leading coefficient, there exists an interval  $(-\infty, \beta)$  on which  $b$  is convex. Thus on this interval,  $b'$  is an increasing function with a well-defined inverse function  $\eta \mapsto \omega(\eta)$ . We claim that for any  $L > 0$  with  $-L \leq \beta$  there exists  $m$  such that for  $\eta < m$ ,  $\omega(\eta) = \lambda(\eta)$ . Indeed, since  $b'$  is an odd-degree polynomial with positive leading coefficient, the number  $m = \inf\{b'(\omega) : \omega \geq -L\}$  is finite. If  $\eta < m$ , the only solution to  $b'(\omega) = \eta$  on  $\mathbb{R}$  must lie in  $(-\infty, -L) \subseteq (-\infty, \beta)$ . Since  $\lambda(\eta)$  is a solution, it lies in this interval. Thus for  $\eta < m$ ,  $\lambda(\eta) = \omega(\eta)$ .

Note that  $b'(\omega) = \omega^{2n-1} + \sum_{j=2}^{2n-1} j a_j \omega^{j-1}$ . Take  $L > 0$  so that  $|\omega| \geq L$  implies

$$\sum_{j=2}^{2n-1} j |a_j| |\omega|^{-2n+j} \leq \frac{1}{2}.$$

Then for  $\omega \leq -L$ ,

$$\frac{3}{2} \omega^{2n-1} \leq \omega^{2n-1} \left( 1 + \sum_{j=2}^{2n-1} j |a_j| |\omega|^{-2n+j} \right) = \omega^{2n-1} - \sum_{j=2}^{2n-1} j |a_j| |\omega|^{j-1} \leq b'(\omega).$$

Since for  $\eta < m$  the solution to  $b'(\omega) = \eta$  is  $\lambda(\eta)$ , this shows that  $\lambda(\eta) \rightarrow -\infty$  as  $\eta \rightarrow -\infty$ . Furthermore, if  $b'(\omega) = \eta$ ,

$$(3.1) \quad \omega^{2n-1} = \eta - \sum_{j=2}^{2n-1} j a_j \omega^{j-1} \Leftrightarrow 1 = \frac{\eta}{\omega^{2n-1}} - \sum_{j=2}^{2n-1} \frac{j a_j}{\omega^{2n-j}} = \frac{\eta}{\omega^{2n-1}} + o(1)$$

as  $\eta \rightarrow -\infty$ . Thus  $\lambda(\eta)^{2n-1} \sim \eta$  as  $\eta \rightarrow -\infty$ , i.e.,  $\lambda(\eta)^{2n-1} = \eta[1 + o(1)]$  as  $\eta \rightarrow -\infty$ . It follows that  $\lambda(\eta) \sim \eta^{\frac{1}{2n-1}}$  as  $\eta \rightarrow -\infty$ .  $\square$

This allows us immediately to obtain size estimates for  $B_\eta(\lambda(\eta)) = -b^*(\eta)$  for large  $\eta$ .

**Lemma 3.2.**

$$(3.2) \quad b^*(\eta) \sim \left( \frac{2n-1}{2n} \right) \eta^{\frac{2n}{2n-1}} \quad \text{as } |\eta| \rightarrow \infty.$$

*Proof.*

$$\begin{aligned} B_\eta(\lambda(\eta)) &= b(\lambda(\eta)) - \eta\lambda(\eta) \\ &= \frac{1}{2n}\lambda(\eta)^{2n} + \sum_{j=2}^{2n-1} a_j \lambda(\eta)^j - \eta\lambda(\eta). \end{aligned}$$

By Lemma 3.1,

$$\begin{aligned} B_\eta(\lambda(\eta)) &= \frac{1}{2n}\eta^{\frac{2n}{2n-1}}(1+o(1))^{2n} + \sum_{j=2}^{2n-1} a_j \left( \eta^{\frac{1}{2n-1}}(1+o(1)) \right)^j - \eta^{\frac{2n}{2n-1}}(1+o(1)) \\ &= \frac{1}{2n}\eta^{\frac{2n}{2n-1}}(1+o(1)) + \sum_{j=2}^{2n-1} a_j \eta^{\frac{j}{2n-1}}(1+o(1)) - \eta^{\frac{2n}{2n-1}}(1+o(1)) \\ &= \left( \frac{1-2n}{2n} \right) \eta^{\frac{2n}{2n-1}}(1+o(1)) + \sum_{j=2}^{2n-1} a_j \eta^{\frac{j}{2n-1}}(1+o(1)) \\ &= \left( \frac{1-2n}{2n} \right) \eta^{\frac{2n}{2n-1}} \left[ (1+o(1)) + \left( \frac{2n}{1-2n} \right) \left( \sum_{j=2}^{2n-1} a_j \eta^{\frac{j-2n}{2n-1}}(1+o(1)) \right) \right] \\ &= \left( \frac{1-2n}{2n} \right) \eta^{\frac{2n}{2n-1}}(1+o(1)) \end{aligned}$$

as  $|\eta| \rightarrow \infty$ , i.e.,

$$(3.3) \quad B_\eta(\lambda(\eta)) \sim \left( \frac{1-2n}{2n} \right) \eta^{\frac{2n}{2n-1}}$$

as  $|\eta| \rightarrow \infty$ . By our definition of  $b^*$ , the result is established.  $\square$

We will also need asymptotic estimates for  $b^{(j)}(\lambda(\eta))$ :

**Lemma 3.3.** For  $j = 2, \dots, 2n$ ,

$$(3.4) \quad b^{(j)}(\lambda(\eta)) \sim \frac{(2n-1)!}{(2n-j)!} \eta^{\frac{2n-j}{2n-1}} \quad \text{as } |\eta| \rightarrow \infty.$$

*Proof.* The proof is similar to that for Lemma 3.2 and is omitted.  $\square$

We close this section with a proposition stating several properties of  $b^*$ .

**Proposition 3.4.** For  $b$  as in (1.2),  $b^*(\eta) = \sup_x [\eta x - b(x)]$  is finite and convex on  $\mathbb{R}$ . It is therefore continuous.

*Proof.* We merely sketch the proof since these are known properties of the Legendre transform. The finiteness of  $b^*$  comes from the fact that  $x \mapsto \eta x - b(x)$  is a non-constant polynomial with even degree and negative leading coefficient. The convexity comes from the fact that  $b^*$  is the supremum of a family  $\{\eta \mapsto \eta x - b(x) : x \in \mathbb{R}\}$  of convex functions. Furthermore,  $b^*$  is continuous since every (finite) convex function is continuous.  $\square$

#### 4. LOCAL PROPERTIES OF $\lambda(\eta)$ AND $B_\eta$

The main result of this section describes those points in  $\mathbb{R}$  that can be (global) minimizers of one of the members of the family of polynomials  $\{B_\eta(\lambda) := -\lambda\eta + b(\lambda) : \eta \in \mathbb{R}\}$ .

**Definition 4.1.** For each  $\eta \in \mathbb{R}$ , define  $\Lambda_\eta$  to be the set of all points at which the polynomial  $B_\eta$  attains its global minimum. Let  $\sigma(\eta)$  be the smallest element of  $\Lambda_\eta$  and let  $\lambda(\eta)$  be the largest. Let  $\mathcal{C} = \{\eta : |\Lambda_\eta| > 1\}$ . Finally, let  $\Lambda = \bigcup_\eta \Lambda_\eta$  and  $\lambda[\mathbb{R}] = \{\lambda(\eta) : \eta \in \mathbb{R}\}$ .

**Theorem 4.2.**  $\lambda[\mathbb{R}] = \mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma(c), \lambda(c))$ .

In the case of a *convex* polynomial  $b$ ,  $b'$  is one-to-one and hence  $B_\eta$  has precisely one critical point for each  $\eta$ . Thus in the convex case,  $\mathcal{C} = \emptyset$  and  $b'$  and  $\lambda$  are inverses. These statements are not true in the non-convex case, though there are partial analogues.

Since all elements of  $\Lambda_\eta$  are solutions to  $\eta = b'(\lambda)$ , the following is immediate.

**Corollary 4.3.**  $\eta \mapsto \lambda(\eta)$  is injective.

We easily verify several other properties of  $\lambda(\cdot)$  and  $b'$ .

**Lemma 4.4.** If  $\lambda_1, \lambda_2 \in \Lambda$  with  $\lambda_1 < \lambda_2$ , then  $b'(\lambda_1) < b'(\lambda_2)$ .

*Proof.* Since  $\lambda_i \in \Lambda$ , there exist  $\eta_1 \neq \eta_2$  such that  $\lambda_1 = \lambda(\eta_1)$  and  $\lambda_2 = \lambda(\eta_2)$ . Since  $\eta_i = b'(\lambda(\eta_i)) = b'(\lambda_i)$ , we must show that  $\eta_1 < \eta_2$ .

Suppose, on the contrary, that  $\eta_2 < \eta_1$ . Since  $\lambda(\eta_i)$  is a point at which  $B_{\eta_i}(\lambda) = -\eta_i\lambda + b(\lambda)$  attains its global minimum,  $B_{\eta_2}(\lambda_2) < B_{\eta_2}(\lambda_1)$ . If  $\eta_2 < \eta_1$ ,

$$\begin{aligned} B_{\eta_1}(\lambda_2) - B_{\eta_1}(\lambda_1) &= -\eta_1\lambda_2 + b(\lambda_2) - (-\eta_1\lambda_1 + b(\lambda_1)) \\ &= (\lambda_1 - \lambda_2)[\eta_2 + (\eta_1 - \eta_2)] + b(\lambda_2) - b(\lambda_1) \\ &= (\lambda_1 - \lambda_2)(\eta_1 - \eta_2) + B_{\eta_2}(\lambda_2) - B_{\eta_2}(\lambda_1) < 0. \end{aligned}$$

This contradicts the fact that  $B_{\eta_1}$  takes its global minimum at  $\lambda_1$  and proves the result.  $\square$

**Lemma 4.5.**  $\lambda : \mathbb{R} \rightarrow \lambda[\mathbb{R}]$  and  $b' : \lambda[\mathbb{R}] \rightarrow \mathbb{R}$  are inverses.

*Proof.* We have already observed that  $\eta = b'(\lambda(\eta))$  for all  $\eta \in \mathbb{R}$ .

Thus consider  $\omega \in \lambda[\mathbb{R}]$ . There exists a unique  $\nu$  such that  $\omega = \lambda(\nu)$ . Since  $\nu = b'(\lambda(\nu)) = b'(\omega)$ , we have  $\omega = \lambda(b'(\omega))$ , as desired.  $\square$

**Corollary 4.6.**  $\lambda : \mathbb{R} \rightarrow \lambda[\mathbb{R}]$  is increasing.

The proof of Theorem 4.2 requires a number of additional technical lemmas.

**Lemma 4.7.** *Take  $c \in \mathcal{C}$ . If  $\omega \in (\sigma(c), \lambda(c)) \setminus \Lambda_c$ , then there does not exist an  $\eta$  for which  $\omega \in \Lambda_\eta$ .*

*Proof.* Since  $\omega$  is not a location of the global minimum of  $B_c$ ,

$$-\omega c + b(\omega) > -\sigma(c)c + b(\sigma(c)) \quad \text{and} \quad -\omega c + b(\omega) > -\lambda(c)c + b(\lambda(c)).$$

Since  $\sigma(c) < \omega < \lambda(c)$ , if  $\eta > c$ ,

$$\begin{aligned} B_\eta(\omega) - B_\eta(\lambda(c)) &= -\eta\omega + b(\omega) + \eta\lambda(c) - b(\lambda(c)) \\ &= (-\omega + \lambda(c))[c + (\eta - c)] + b(\omega) - b(\lambda(c)) \\ &= B_c(\omega) - B_c(\lambda(c)) + [\lambda(c) - \omega][\eta - c] > 0. \end{aligned}$$

Similarly, for  $\eta < c$ ,  $B_\eta(\omega) - B_\eta(\sigma(c)) > 0$ . We conclude that there is no  $\eta \in \mathbb{R}$  for which  $\omega$  is the location of the global minimum of  $B_\eta$ .  $\square$

**Corollary 4.8.** *If  $\eta_1 \neq \eta_2$ , then  $\Lambda_{\eta_1} \cap \Lambda_{\eta_2} = \emptyset$ . Furthermore, if  $c_1, c_2 \in \mathcal{C}$  with  $c_1 \neq c_2$ ,  $[\sigma(c_1), \lambda(c_1)) \cap [\sigma(c_2), \lambda(c_2)) = \emptyset$ .*

**Lemma 4.9.** *Let  $\deg b = 2n$ . Then  $|\mathcal{C}| \leq n - 1$ .*

*Proof.* Let  $c \in \mathcal{C}$ . Then  $(\sigma(c), \lambda(c))$  is non-empty. Since  $\lambda \mapsto -c\lambda + b(\lambda)$  takes the same value at  $\sigma(c)$  and  $\lambda(c)$ , by Rolle's theorem, there exists  $\omega_0 \in (\sigma(c), \lambda(c))$  at which  $-c + b'(\omega_0) = 0$ , i.e.,  $\omega_0$  is another critical point of  $B_c$ . Since  $-c + b'(\sigma(c)+) > 0$  but  $-c + b'(\lambda(c)-) < 0$ , we may take the point  $\omega_0$  to be a local maximum of  $B_c$ . Thus  $B_c'' = b''$  must change sign in each of  $(\sigma(c), \omega_0)$  and  $(\omega_0, \lambda(c))$ . Since by Corollary 4.8 the intervals in the collection  $\{(\sigma(c), \lambda(c)) : c \in \mathcal{C}\}$  are disjoint, the total number  $|\mathcal{C}|$  can not exceed  $\frac{1}{2} \deg b'' = n - 1$ .  $\square$

The next lemma is central, as it identifies subintervals of  $\mathbb{R}$  in which  $\lambda[\mathbb{R}]$  is dense.

**Lemma 4.10.** *Let  $\alpha, \eta_0 \in \mathbb{R}$ .*

- (1) *If  $a < \sigma(\eta_0)$ ,  $(a, \sigma(\eta_0)) \cap \lambda[\mathbb{R}] \neq \emptyset$ .*
- (2) *If  $\lambda(\eta_0) < a$ ,  $(\lambda(\eta_0), a) \cap \lambda[\mathbb{R}] \neq \emptyset$ .*

*Proof.* We note that  $\eta_0$  need not be an element of  $\mathcal{C}$ . If it is not,  $\sigma(\eta_0) = \lambda(\eta_0)$ .

We prove the first statement. The proof of the second is similar. If  $\omega < \sigma(\eta_0)$ , then

$$(4.1) \quad B_{\eta_0}(\omega) > B_{\eta_0}(\sigma(\eta_0)).$$

Fix  $a < \sigma(\eta_0)$ . Since  $B_{\eta_0}$  is continuous, for any  $L > 0$  satisfying  $-L < a$ , there exists  $d$  (depending on  $L$ ) such that for all  $\omega \in [-L, a]$ ,

$$B_{\eta_0}(\omega) \geq d > B_{\eta_0}(\sigma(\eta_0)) \iff B_{\eta_0}(\omega) - B_{\eta_0}(\sigma(\eta_0)) \geq d - B_{\eta_0}(\sigma(\eta_0)) := \alpha > 0.$$

We choose  $L$  as follows: Since, by Lemma 3.1,  $\lambda(\eta) \rightarrow -\infty$  as  $\eta \rightarrow -\infty$ , there exists  $\eta^* < \eta_0 - 1$  satisfying  $\lambda(\eta^*) < -|a| - 1$ . Set  $-L := \lambda(\eta^*)$ .

Set  $\varepsilon = \min\{1, \frac{\alpha}{2\sigma(\eta_0)+L}\}$ . We claim that for all  $\eta \in (\eta_0 - \varepsilon, \eta_0)$  and  $\omega \in [-L, a]$ ,

$$(4.2) \quad B_\eta(\omega) > B_\eta(\sigma(\eta_0)).$$



Indeed, since  $B_\eta(\omega) = B_{\eta_0}(\omega) - (\eta - \eta_0)\omega$  and  $B_\eta(\sigma(\eta_0)) = B_{\eta_0}(\sigma(\eta_0)) - (\eta - \eta_0)\sigma(\eta_0)$ ,

$$\begin{aligned} B_\eta(\omega) - B_\eta(\sigma(\eta_0)) &= B_{\eta_0}(\omega) - B_{\eta_0}(\sigma(\eta_0)) + (\eta - \eta_0)(\sigma(\eta_0) - \omega) \\ &\geq \alpha - \varepsilon(\sigma(\eta_0) - \omega) \\ &\geq \alpha - \frac{\alpha}{2(\sigma(\eta_0) + L)}(\sigma(\eta_0) + L) = \frac{\alpha}{2}. \end{aligned}$$

This proves (4.2).

Finally, we claim that if  $\eta \in (\eta_0 - \varepsilon, \eta_0)$ , then  $\lambda(\eta) \in (a, \sigma(\eta_0))$ . Since  $\eta < \eta_0$  and  $\lambda$  is an increasing function,  $\lambda(\eta) < \lambda(\eta_0)$ . By Lemma 4.7 this forces  $\lambda(\eta) < \sigma(\eta_0)$ . Since  $B_\eta(\lambda(\eta)) < B_\eta(\sigma(\eta_0))$ , by (4.2),  $\lambda(\eta) \notin [-L, a]$ . If  $\lambda(\eta)$  were less than  $-L = \lambda(\eta^*)$ , then the fact that  $\lambda$  is increasing would imply  $\eta < \eta^* < \eta_0 - 1$ , which is false. We conclude that  $\lambda(\eta) \in (a, \sigma(\eta_0))$ .  $\square$

We are now ready to prove Theorem 4.2.

*Proof.* That  $\lambda[\mathbb{R}] \subseteq \mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma(c), \lambda(c))$  follows from Lemma 4.7 and Corollary 4.8.

Next we prove  $\mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma(c), \lambda(c)) \subseteq \lambda[\mathbb{R}]$ . Suppose  $\mathcal{C} \neq \emptyset$ . Since  $|\mathcal{C}|$  is finite, we may order the elements of  $\mathcal{C}$  so that  $c_i < c_{i+1}$ ,  $1 \leq i \leq k-1$ . The left-hand set is made up of three kinds of intervals: two semi-infinite intervals  $(-\infty, \sigma(c_1))$  and  $[\lambda(c_k), \infty)$ , and (if  $k \geq 2$ ) the intervals  $[\lambda(c_i), \sigma(c_{i+1}))$ . We must show that every  $\omega$  in one of these intervals is in  $\lambda[\mathbb{R}]$ .

Consider first the case in which  $k \geq 2$  and  $\omega \in (\lambda(c_i), \sigma(c_{i+1}))$ . Set

$$U := (\lambda(c_i), \omega) \cap \lambda[\mathbb{R}], \quad V := (\omega, \sigma(c_{i+1})) \cap \lambda[\mathbb{R}], \quad \nu := \inf \{ b'(\lambda) : \lambda \in V \}.$$

By Lemma 4.10,  $V \neq \emptyset$  and hence  $\nu$  is defined. We claim  $\omega = \lambda(\nu)$ .

If  $\lambda(\eta) \in V$ , then  $\lambda(\eta) < \sigma(c_{i+1}) < \lambda(c_{i+1})$ , and the monotonicity of  $b'$  on  $\lambda[\mathbb{R}]$  implies

$$\nu \leq b'(\lambda(\eta)) < b'(\lambda(c_{i+1})) = c_{i+1}.$$

Furthermore, since  $U$  contains some  $\lambda(\eta_0)$ , for  $\lambda(\eta) \in V$ ,  $\lambda(c_i) < \lambda(\eta_0) < \omega < \lambda(\eta)$ , so that  $c_i < \nu$ . It follows from the monotonicity of  $\lambda(\cdot)$  that  $\lambda(c_i) < \lambda(\nu) < \sigma(c_{i+1})$ . Thus either  $\lambda(\nu) \in U$ ,  $\lambda(\nu) \in V$ , or  $\lambda(\nu) = \omega$ .

Suppose  $\lambda(\nu) \in U$ . By Lemma 4.10,  $(\lambda(\nu), \omega) \cap \lambda[\mathbb{R}]$  is not empty. It thus contains  $\lambda(\eta_0)$  for some  $\eta_0 > \nu$ . But then  $\eta_0$  would be a lower bound for  $\{ b'(\lambda) : \lambda \in V \}$ , contradicting the definition of  $\nu$ . Thus  $\lambda(\nu) \notin U$ .

Suppose  $\lambda(\nu) \in V$ . Since  $\nu \notin \mathcal{C}$ ,  $\lambda(\nu) = \sigma(\nu)$ . Consider  $(\omega, \sigma(\nu)) \cap \lambda[\mathbb{R}]$ . By Lemma 4.10, this is not empty, and thus there exists  $\lambda(\eta_0)$  in this set, hence in  $V$ , with  $\eta_0 = b'(\lambda(\eta_0)) < b'(\lambda(\nu)) = \nu$ . This contradicts the fact that  $\nu$  is a lower bound for  $\{ b'(\lambda) : \lambda \in V \}$ . Thus  $\lambda(\nu) \notin V$ . We conclude that  $\lambda(\nu) = \omega$ .

The proof in the case of the semi-infinite intervals  $(-\infty, \sigma(c_1))$  and  $(\lambda(c_k), \infty)$  is virtually identical. If  $|\mathcal{C}| = 0$ , one can take  $\sigma(c_1) (= \lambda(c_1))$  arbitrarily large since  $\lambda(\eta) \rightarrow \infty$  as  $\eta \rightarrow \infty$  to conclude that  $\lambda[\mathbb{R}] = \mathbb{R}$ .  $\square$

This theorem, together with Lemma 4.9, yields the following:

**Corollary 4.11.**  *$b$  is convex on  $\mathbb{R} \setminus \bigcup_{c \in \mathcal{C}} [\sigma(c), \lambda(c))$ .*

5. ESTIMATES FOR  $\int_{-\infty}^{\infty} e^{-2\tau p_{\eta}(\xi)} d\xi$ 

Recall that  $p_{\eta}(\xi) = -\eta\xi + b(\xi + \lambda(\eta)) - b(\lambda(\eta))$ . In what follows, we will sometimes suppress the dependence of  $p$  and  $\lambda$  on  $\eta$ . Within this section, we define

$$(5.1) \quad I := \int_{-\infty}^{\infty} e^{-2\tau p_{\eta}(\xi)} d\xi.$$

In the paper [GHar], we obtained sharp estimates on integrals of the form  $I$  that are uniform in the coefficients of  $p$  *under the hypothesis* that  $p$  has degree four. We showed there that those estimates do not generalize to polynomials of higher degree. The estimates obtained below are less precise but are nonetheless sufficient to prove our results on absolute convergence of the integral defining Szegő kernel.

**5.1. Estimates for  $I$  for convex  $p$ .** As in the fourth-degree setting, our analysis makes use of known results on the integral of  $e^{-p}$  over intervals on which  $p$  is convex. Such results do *not* require  $p$  to have degree four. We recall the main result here:

**Lemma 5.1** (Lemma 4.9, [GHar]). *Let  $n$  be a positive integer and define  $p(\xi) = \sum_{j=2}^{2n} \beta_j \xi^j$ . Suppose  $p$  is convex on  $J$ , where  $J$  is one of the intervals  $(-\infty, \infty)$ ,  $(0, \infty)$ , or  $(-\infty, 0)$ . Then*

$$(5.2) \quad \int_J e^{-p(\xi)} d\xi \approx \left[ \sum_{j=2}^{2n} |\beta_j|^{\frac{1}{j}} \right]^{-1}.$$

This lemma follows from work of Bruna, Nagel, and Wainger [BNW88]. A more detailed discussion, including variations and proofs, can be found in Section 4.2 of [GHar].

**5.2. A lower bound for  $I$  for non-convex  $p$ .** By construction,  $p_{\eta}$  vanishes to at least second order at the origin. Thus

$$p_{\eta}(\xi) = \sum_{j=2}^{2n} \frac{p^{(j)}(0)}{j!} \xi^j = \sum_{j=2}^{2n} \frac{b^{(j)}(\lambda(\eta))}{j!} \xi^j \leq \frac{1}{2} \sum_{j=2}^{2n} |b^{(j)}(\lambda(\eta))| |\xi|^j,$$

and (suppressing the dependence of  $\lambda$  on  $\eta$ )

$$(5.3) \quad \begin{aligned} I &\geq \int_{-\infty}^{\infty} e^{-\tau \sum_{j=2}^{2n} |b^{(j)}(\lambda)| |\xi|^j} d\xi \\ &\geq \int_0^{\infty} e^{-\sum_{j=2}^{2n} \tau |b^{(j)}(\lambda)| \xi^j} d\xi \\ &\approx \left[ \sum_{j=2}^{2n} \tau^{\frac{1}{j}} |b^{(j)}(\lambda)|^{\frac{1}{j}} \right]^{-1}, \end{aligned}$$

where in the last line we have used Lemma 5.1 applied to  $\xi \mapsto \sum_{j=2}^{2n} \tau |b^{(j)}(\lambda)| \xi^j$ , which is clearly a convex polynomial on  $(0, \infty)$ .

**5.3. An upper bound for  $I$  for non-convex  $p$ .** An upper bound for  $I$  will give rise to a lower bound for the factor  $[N(\eta, \tau)]^{-1}$  appearing in the integrand for the Szegő kernel. These estimates are therefore necessary for the results on the divergence of the integral  $S[(x, y, t), (r, y, t)]$ . We will see in Section 6 that for  $M$  sufficiently large, the contribution to  $S$  from  $\{(\eta, \tau) : |\eta| > M, \tau > 0\}$  is finite for any  $x$  and  $r$ . Thus when  $S[(x, y, t), (r, y, t)]$  is divergent, it is because there exists some finite  $\eta_0$  for which the contribution to  $S$  from  $\{(\eta, \tau) : \eta_0 < \eta < \eta_0 + \varepsilon, \tau > 0\}$  is infinite. The following proposition is therefore sufficient to establish these results.

**Proposition 5.2.** *Fix  $\eta_0 \in \mathbb{R}$  and  $\varepsilon > 0$ . Then there exists  $c := c(\eta_0, \varepsilon)$  such that for all  $\eta \in (\eta_0, \eta_0 + \varepsilon)$  and for all  $\tau > 0$ ,*

$$(5.4) \quad I \leq c \frac{1 + \tau^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}.$$

We begin by factoring  $p_\eta$ . For fixed  $\eta$ , this is a non-negative polynomial vanishing to even order at the origin, its real roots are of even multiplicity, and its non-real roots occur in complex conjugate pairs. Its factorization over  $\mathbb{C}$  may therefore be written

$$(5.5) \quad p_\eta(\xi) = \frac{1}{2n} \xi^2 \prod_{j=2}^n [\xi - \alpha_j(\eta)][\xi - \overline{\alpha_j(\eta)}],$$

where the  $\alpha_j$  may be real and need not be distinct. Furthermore, if  $\alpha_j(\eta) = h_j(\eta) + ik_j(\eta)$ , we order the roots so that  $h_2(\eta) \leq h_3(\eta) \leq \dots \leq h_n(\eta)$ . The factorization of  $p_\eta$  over  $\mathbb{R}$  is thus

$$(5.6) \quad p_\eta(\xi) = \frac{1}{2n} \xi^2 \prod_{j=2}^n [(\xi - h_j(\eta))^2 + k_j^2(\eta)].$$

In what follows, we denote the  $j$ -th quadratic factor in the above product by  $q_j(\xi, \eta)$ .

Since the  $h_j$  are functions of  $\eta$  and we seek estimates for  $I$  that are valid for all  $\eta$  throughout an interval, we need a lemma on the local behavior of the  $h_j$ :

**Lemma 5.3.** *Fix  $\eta_0 \in \mathbb{R}$  and  $\varepsilon > 0$ . Then there exists  $C > 1$  such that for all  $\eta \in J := [\eta_0, \eta_0 + \varepsilon]$  and for all  $j$ ,  $|h_j(\eta)| \leq C - 1$ .*

*Proof.* This is a standard argument. Suppose the result fails, so that for some  $j$ ,  $h_j$  is unbounded on  $J$ . Assume without loss of generality that  $j = 2$ . Let  $(\eta_\ell)$  be a sequence in  $J$  for which  $|h_2(\eta_\ell)| \rightarrow \infty$ . By extracting subsequences if necessary, we may assume that  $(\eta_\ell)$  converges to some  $\eta' \in J$ .

Recall that, by definition,  $p_\eta(\xi) = -\eta(\xi + \lambda(\eta)) + b(\xi + \lambda(\eta)) - b^*(\eta)$ . Since  $b, b^*$  are continuous and  $\lambda$  is bounded on  $J$ , for  $\xi$  fixed, there exists  $M_\xi > 0$  such that for all  $\eta \in J$ ,  $0 \leq p_\eta(\xi) \leq M_\xi$ . Fix  $\xi = 1$ . Then for all  $\ell$ ,

$$0 \leq \frac{1}{2n} \prod_{j=2}^n q_j(1, \eta_\ell) \leq M_1.$$

Since  $\lim_{\ell \rightarrow \infty} q_2(1, \eta_\ell) = \infty$ ,  $\lim_{\ell \rightarrow \infty} \prod_{j=3}^n q_j(1, \eta_\ell) = 0$ . Thus there exists a factor  $q_{j_1}$  and a subsequence  $(\eta_\ell^{(1)})$  such that  $\lim_{\ell \rightarrow \infty} q_{j_1}(1, \eta_\ell^{(1)}) = 0$ . This forces  $\lim_{\ell \rightarrow \infty} h_{j_1}(\eta_\ell^{(1)}) = 1$ .

Now take  $\xi = 2$ . It is still the case that  $\lim_{\ell \rightarrow \infty} q_2(2, \eta_\ell^{(1)}) = \infty$ , but now for  $\ell$  sufficiently large,  $q_{j_1}(2, \eta_\ell^{(1)})$  is bounded away from zero. These facts, together

with the boundedness of  $p_\eta(2)$  on  $J$ , allow us to find a different factor  $q_{j_2}$  and a subsequence  $(\eta_\ell^{(2)})$  of  $(\eta_\ell^{(1)})$  such that  $q_{j_2}(2, \eta_\ell^{(2)}) \rightarrow 0$  and  $h_{j_2}(\eta_\ell^{(2)}) \rightarrow 2$ . Repeating this process at most  $n - 2$  times leads to a subsequence  $(\nu_\ell)$  of the original such that  $h_{j_i}(\nu_\ell)$  tends to  $i$ .

Fix  $\xi = n$ . There exists  $M_n$  such that  $0 \leq p_\eta(n) \leq M_n$  for all  $\eta \in J$ . Furthermore, each  $q_j(n, \nu_\ell)$ ,  $2 \leq j \leq n$  is bounded away from zero for  $\ell$  sufficiently large, but  $q_1(n, \nu_\ell)$  is unbounded. This is a contradiction, and the lemma is proved.  $\square$

We now prove Proposition 5.2. With notation as in the proof of Lemma 5.3, we find

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^n [(\xi - h_j)^2 + k_j^2] \right) d\xi \\ &\leq \int_{-\infty}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^n (\xi - h_j)^2 \right) d\xi. \end{aligned}$$

Write this last integral as the sum of integrals  $I_1$ ,  $I_2$ , and  $I_3$ , where  $I_1$  is over the interval  $(-\infty, C)$ ,  $I_2$  is over  $[-C, C]$ , and  $I_3$  is over  $(C, \infty)$ .

Since  $-(C - 1) \leq h_j(\eta) \leq C - 1$  for each  $j$  and for all  $\eta \in J$ ,

$$\begin{aligned} I_1 &\leq \int_{-\infty}^{-C} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^n (-C - h_j)^2 \right) d\xi \\ &\leq \int_{-\infty}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^n (-C - h_j)^2 \right) d\xi \\ &\approx \left( \frac{\tau}{n} \prod_{j=2}^n (C + h_j)^2 \right)^{-\frac{1}{2}} \\ &\approx \tau^{-\frac{1}{2}} \end{aligned}$$

since  $2C - 1 \geq C + h_j(\eta) \geq 1$  for all  $j$  and for all  $\eta \in J$ .

We make the simplest possible estimate of  $I_2$ ; since the integrand is less than 1,

$$I_2 \leq 2C \approx 1.$$

We estimate  $I_3$  in the same way as  $I_1$ , using now the fact that for all  $j$ ,  $h_j(\eta) < C - 1$  for all  $\eta \in J$ .

$$\begin{aligned}
I_3 &\leq \int_C^\infty \exp\left(-\frac{\tau}{n}\xi^2 \prod_{j=2}^n (C - h_j)^2\right) d\xi \\
&\leq \int_{-\infty}^\infty \exp\left(-\frac{\tau}{n}\xi^2 \prod_{j=2}^n (C - h_j)^2\right) d\xi \\
&\approx \left(\frac{\tau}{n} \prod_{j=2}^n (C - h_j)^2\right)^{-\frac{1}{2}} \\
&\approx \tau^{-\frac{1}{2}}.
\end{aligned}$$

Putting the three estimate together yields  $I \lesssim \max\{\tau^{-\frac{1}{2}}, 1\} \approx \frac{1+\tau^{\frac{1}{2}}}{\tau^{\frac{1}{2}}}$ , as claimed.

## 6. PROOFS OF THEOREMS

If we show that for all non-negative integers  $i_1, j_1, i_2$  and  $j_2$ , each integral

$$\iint_{\tau>0} e^{\eta\tau[z_1+\bar{w}_1]+i\tau[z_2-\bar{w}_2]} \frac{\eta^{i_1+j_1} \tau^{i_1+j_1+i_2+j_2+1}}{N(\eta, \tau)} d\eta d\tau$$

is absolutely convergent in the region in which

$$h + k + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0,$$

it will follow that this integral is in fact equal to  $\partial_{z_1}^{i_1} \partial_{\bar{w}_1}^{j_1} \partial_{z_2}^{i_2} \partial_{\bar{w}_2}^{j_2} S(z, w)$ .

Set  $\delta = h + k$ ,  $(z_1, z_2) = (x + iy, t + ib(x) + ih)$ ,  $(w_1, w_2) = (r + is, u + ib(r) + ik)$ ,  $s = i_1 + j_1$ , and  $m = i_1 + j_1 + i_2 + j_2$  (so that  $m \geq s$ ). The integral becomes

$$(6.1) \quad S^{s,m,\delta} := \iint_{\tau>0} e^{\eta\tau[x+r+i(y-s)]+i\tau[t-u+i(b(x)+b(r)+\delta)]} \frac{\eta^s \tau^{m+1}}{N(\eta, \tau)} d\eta d\tau,$$

and it converges absolutely if and only if

$$(6.2) \quad \tilde{S}^{s,m,\delta} := \int_{-\infty}^\infty \int_0^\infty e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)]} \frac{|\eta|^s \tau^{m+1}}{N(\eta, \tau)} d\tau d\eta < \infty.$$

From (5.3),

$$\begin{aligned}
&\tilde{S}^{s,m,\delta} \\
&= \int_{-\infty}^\infty \int_0^\infty e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)]} \frac{|\eta|^s \tau^{m+1}}{e^{2\tau b^*(\eta)} I} d\tau d\eta \\
&\geq \sum_{j=2}^{2n} \int_{-\infty}^\infty \int_0^\infty e^{-\tau[\delta+b(x)+b(r)-\eta(x+r)+2b^*(\eta)]} |\eta|^s \tau^{m+1+\frac{1}{j}} |b^{(j)}[\lambda(\eta)]|^{\frac{1}{j}} d\tau d\eta \\
&:= \sum_{j=2}^{2n} \int_{-\infty}^\infty I_j^{s,m,\delta}(\eta) d\eta \\
&:= \sum_{j=2}^{2n} I_j^{s,m,\delta}.
\end{aligned}$$

Further, set  $A(x, r, \eta) := b(x) + b(r) - \eta(x + r) + 2b^*(\eta)$ . If  $\delta + A(x, r, \eta) > 0$ , we can evaluate the  $\tau$  integral.

$$(6.3) \quad I_j^{s,m,\delta}(\eta) \approx \frac{|\eta|^s |b^{(j)}(\lambda)|^{\frac{1}{j}}}{[\delta + A(x, r, \eta)]^{m+2+\frac{1}{j}}}.$$

We see that there are two possible barriers to the convergence of the integrals  $I_j^{s,m,\delta}$ :

- (1) insufficient decay of  $I_j^{s,m,\delta}(\eta)$  for fixed  $x, r$  as  $|\eta| \rightarrow \infty$ ;
- (2) vanishing of  $\delta + A(x, r, \eta)$  at some finite  $\eta$  for certain choices of  $x, r$ , and  $\delta$ .

We deal with these in turn.

### 6.1. Behavior of $I_j^{s,m,\delta}(\eta)$ for large $|\eta|$ .

**Lemma 6.1.** *Fix  $x, r \in \mathbb{R}$ ,  $\delta > 0$ . Then*

$$\delta + A(x, r, \eta) \sim \left( \frac{2n-1}{n} \right) \eta^{\frac{2n}{2n-1}}, \quad |\eta| \rightarrow \infty.$$

*Proof.* This follows immediately from Lemma 3.2, since

$$\begin{aligned} \delta + A(x, r, \eta) &= \delta + b(x) + b(r) - \eta(x + r) + 2b^*(\eta) \\ &= \left( \frac{2n-1}{n} \right) \eta^{\frac{2n}{2n-1}} (1 + o(1)) \end{aligned}$$

as  $|\eta| \rightarrow \infty$ . □

We may use Lemmas 3.3 and 6.1 to estimate the integrals for large  $|\eta|$ :

$$\begin{aligned} |I_j^{s,m,\delta}(\eta)| &\sim c \frac{|\eta|^s}{\eta^{\frac{2n(m+2)}{2n-1}}} \frac{|\eta|^{\frac{2n-j}{j(2n-1)}}}{\eta^{\frac{2n}{j(2n-1)}}} \\ &= c |\eta|^{-2-(m-s)-\frac{m+3}{2n-1}} \end{aligned}$$

as  $|\eta| \rightarrow \infty$ . Since  $m \geq s \geq 0$ ,  $-2-(m-s)-\frac{m+3}{2n-1} < -2$ . Thus for any fixed  $s, m, j$ , and  $\delta > 0$ ,  $I_j^{s,m,\delta}$  is convergent at infinity.

**6.2. Vanishing of  $\delta + A(x, r, \eta)$ .** In light of the previous section, we see that whether or not one of the integrals  $I_j$  converges depends on whether or not the function  $\eta \mapsto \delta + A(x, r, \eta)$  vanishes at a finite  $\eta_0$  for some fixed  $x, r, \delta$ , and the behavior of this function near such zeros. In fact, we have proved

**Proposition 6.2.** *If for some fixed  $x, r$ , and  $\delta$*

$$(6.4) \quad \inf_{\eta} \delta + A(x, r, \eta) > 0,$$

*then all the  $I_j^{s,m,\delta}$  are finite.*

Furthermore,

$$\begin{aligned} \inf_{\eta} \delta + A(x, r, \eta) &= \delta + b(x) + b(r) - 2 \sup \left[ \eta \left( \frac{x+r}{2} \right) - b^*(\eta) \right] \\ &= \delta + b(x) + b(r) - 2b^{**} \left( \frac{x+r}{2} \right), \end{aligned}$$

where the convexity of  $b^*$  and its super-linear growth at infinity (Lemma 3.2) guarantee the finiteness of the supremum in the first line. It follows that the integrals

defining the Szegő kernel and all of its derivatives converge absolutely in the region in which

$$\delta + b(x) + b(r) - 2b^{**}\left(\frac{x+r}{2}\right) > 0.$$

This is precisely the region defined in (2.10). To prove the remainder of Theorem 2.3, we must use the results of Section 4 to identify points  $(z, w) = [(z_1, z_2), (w_1, w_2)] = [(x + iy, t + i(b(x) + h)), (r + is, u + i(b(r) + k))]$  satisfying (2.10).

If  $(z, w) \in (\Omega \times \bar{\Omega}) \cup (\bar{\Omega} \times \Omega)$ ,  $\delta > 0$ . Since  $A(x, r, \eta) \geq 0$ , such  $(z, w)$  are indeed in the region (2.10). We turn our attention, then, to points  $(z, w) \in \partial\Omega \times \partial\Omega$ , where  $\delta = 0$ .

Set

$$(6.5) \quad A_x(\eta) := b^*(\eta) - [\eta x - b(x)] \quad A_r(\eta) := b^*(\eta) - [\eta r - b(r)],$$

so that

$$(6.6) \quad A(x, r, \eta) = A_x(\eta) + A_r(\eta).$$

Fix  $x$  and  $r$ , and recall the definition of  $\Lambda_{\eta_0}$  from (2.5).  $A(x, r, \eta_0) = 0$  if and only if  $A_x(\eta_0) = A_r(\eta_0) = 0$ . This, in turn, happens precisely when  $x, r \in \Lambda_{\eta_0}$ .

From Lemma 3.2 and Proposition 3.4, it follows that  $A(x, r, \cdot)$  is a continuous function of  $\eta$  which grows at infinity like  $c|\eta|^{\frac{2n}{2n-1}}$ . Thus if for some fixed  $x$  and  $r$  it does not vanish, it is bounded below by a positive constant. Together with Lemma 6.1 this shows that if  $(z, w) \in (\partial\Omega \times \partial\Omega) \setminus \Sigma$ , the integrals defining the Szegő kernel and all its derivatives are absolutely convergent.

Finally, we turn to the proof of Theorem 2.5. We must consider the integrals  $S^{0,0,\delta}$  and  $\tilde{S}^{0,0,\delta}$  from (6.1) and (6.2). To simplify notation, we drop the additional superscripts. We must show

- (i)  $\tilde{S}^0$  is divergent, and
- (ii)  $\lim_{\delta \rightarrow 0^+} \tilde{S}^\delta = \infty$ ,

whenever there exists  $\eta_0$  such that  $x, r \in \Lambda_{\eta_0}$ . Clearly (i) implies (ii) since the integrand of  $\tilde{S}^\delta$  is non-negative and converges pointwise and monotonically to the integrand of  $\tilde{S}^0$  as  $\delta \rightarrow 0^+$ . We thus consider (i).

Fix  $\eta_0 \in \mathbb{R}$ ,  $\varepsilon > 0$ , and  $x, r \in \Lambda_{\eta_0}$ . Applying Proposition 5.2,

$$(6.7) \quad \frac{\tau^{\frac{1}{2}}}{1 + \tau^{\frac{1}{2}}} \lesssim e^{2\tau b^*(\eta)} N(\eta, \tau)^{-1},$$

for all  $\tau > 0$  and  $\eta \in (\eta_0, \eta_0 + \varepsilon)$ .

Substituting into (6.2) and recalling the definition of  $A(= A(x, r, \eta))$  from (6.6) gives

$$(6.8) \quad \begin{aligned} \tilde{S}^0 &> \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^\infty \tau e^{-\tau A} e^{2\tau b^*(\eta)} N(\eta, \tau)^{-1} d\tau d\eta \\ &\gtrsim \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^\infty \frac{\tau^{\frac{3}{2}} e^{-\tau A}}{1 + \tau^{\frac{1}{2}}} d\tau d\eta \\ &= \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^\infty \frac{e^{-\tau} \tau^{\frac{3}{2}}}{A^2 [A^{\frac{1}{2}} + \tau^{\frac{1}{2}}]} d\tau d\eta. \end{aligned}$$

It is now clear that we need a lemma on the order of vanishing of  $A(x, r, \eta)$  at  $\eta_0$ .

**Lemma 6.3.** *Take  $b$  as in (1.2),  $\eta_0 \in \mathbb{R}$ , and  $x \in \Lambda_{\eta_0}$ . Then*

$$A_x(\eta) = (\eta - \eta_0)F_x(\eta)$$

for all  $\eta \in (\eta_0, \infty)$ , where  $F_x$  is bounded on each interval  $(\eta_0, \eta_0 + \varepsilon)$ .

*Proof.* Since  $\lambda(\eta_0)$  is the largest element of  $\Lambda_{\eta_0}$ ,  $\lambda(\eta_0) \geq x$ . Since  $\lambda(\cdot)$  injective and increasing, for any  $\eta > \eta_0$ ,  $\lambda(\eta) > x$ . Thus for  $\eta > \eta_0$ ,

$$\begin{aligned} A_x(\eta) &= b(x) - \eta x + b^*(\eta) \\ &= b(x) - \eta x + \eta \lambda(\eta) - b(\lambda(\eta)) \\ &= \eta(\lambda(\eta) - x) + b(x) - b(\lambda(\eta)) - \eta_0(\lambda(\eta) - x) + \eta_0(\lambda(\eta) - x) \\ &= (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x) \left[ \frac{b(\lambda(\eta)) - b(x) - \eta_0(\lambda(\eta) - x)}{\lambda(\eta) - x} \right] \\ &= (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x)\phi_x(\eta). \end{aligned}$$

Observe,

$$\phi_x(\eta) = \frac{b(\lambda(\eta)) - \eta_0 \lambda(\eta) - [b(x) - \eta_0 x]}{\lambda(\eta) - x} = \frac{B_{\eta_0}(\lambda(\eta)) - B_{\eta_0}(x)}{\lambda(\eta) - x}.$$

Since  $x \in \Lambda_{\eta_0}$ , the minimality of  $B_{\eta_0}(x)$  yields  $B_{\eta_0}(\lambda(\eta)) \geq B_{\eta_0}(x)$  for all  $\eta \in \mathbb{R}$ . Therefore  $\phi_x$  is non-negative on the interval  $(\eta_0, \infty)$ . It follows that for  $\eta > \eta_0$ ,

$$(6.9) \quad A_x(\eta) = (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x)\phi_x(\eta) \geq 0 \iff 1 \geq \frac{\phi_x(\eta)}{\eta - \eta_0}.$$

Hence, on  $(\eta_0, \infty)$ ,

$$\begin{aligned} A_x(\eta) &= (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x)\phi_x \\ &= (\eta - \eta_0)(\lambda(\eta) - x) \left[ 1 - \frac{\phi_x(\eta)}{\eta - \eta_0} \right] \\ &:= (\eta - \eta_0)F_x(\eta). \end{aligned}$$

By inequality (6.9) and the local boundedness of  $\lambda(\eta)$ ,  $F_x$  is bounded on each interval  $(\eta_0, \eta_0 + \varepsilon)$ . This proves the lemma.  $\square$

We use this lemma to substitute for  $A$  in (6.8)

$$\begin{aligned} \tilde{S}^0 &\geq \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^1 \frac{e^{-\tau} \frac{\tau^{\frac{3}{2}}}{(\eta - \eta_0)^2 (F_x(\eta) + F_r(\eta))^2}}{(\eta - \eta_0)^{\frac{1}{2}} (F_x(\eta) + F_r(\eta))^{\frac{1}{2}} + 1} d\tau d\eta \\ &= \left( \int_0^1 \tau^{\frac{3}{2}} e^{-\tau} d\tau \right) \int_{\eta_0}^{\eta_0 + \varepsilon} \frac{\frac{1}{(\eta - \eta_0)^2 (F_x(\eta) + F_r(\eta))^2}}{(\eta - \eta_0)^{\frac{1}{2}} (F_x(\eta) + F_r(\eta))^{\frac{1}{2}} + 1} d\eta \\ &\approx \int_{\eta_0}^{\eta_0 + \varepsilon} \frac{G(\eta)}{(\eta - \eta_0)^2 (F_x(\eta) + F_r(\eta))^2} d\eta, \end{aligned}$$

where  $G(\eta)$  is right-continuous and bounded away from zero. Since  $F_x + F_r$  is also a locally-bounded positive function, the divergence of the integral follows. This completes the proof of Theorem 2.5.



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